

Gauge conservation laws in a general setting. Superpotential

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The fact that the conserved current of a gauge symmetry is reduced to a superpotential is proved in a very general setting.

I. INTRODUCTION

The fact that the conserved current of a gauge symmetry is reduced to a superpotential has been stated in different particular variants, e.g., gauge theory of principal connections and gauge gravitation theory.^{1–4} We aim to prove this assertion in a very general setting.

Generic higher-order Lagrangian theory of even and odd fields on an n -dimensional smooth manifold X and its variational generalized supersymmetries (henceforth symmetries) are considered.^{5–8} These symmetries form a real vector space \mathcal{G}_L . In a general setting, a gauge symmetry of a Lagrangian L is defined as a \mathcal{G}_L -valued linear differential operator on some Grassmann-graded projective $C^\infty(X)$ -module of finite rank.^{7,8} Note that any Lagrangian possesses gauge symmetries which therefore must be separated into the trivial and non-trivial ones. However, there is a problem of defining non-trivial gauge symmetries.⁷

In contrast with gauge symmetries, non-trivial Noether identities of Lagrangian field theory are well described in homology terms.^{6–8} Therefore, we define non-trivial gauge symmetries as those associated to complete non-trivial Noether identities in accordance with the second Noether theorem (Theorem 5).

Given a non-trivial gauge symmetry of a Lagrangian L , the corresponding current \mathcal{J} (12) is conserved by virtue of the first Noether theorem (Theorem 3). We prove that this current takes the superpotential form

$$\mathcal{J}^\mu = W^\mu + d_\nu U^{\nu\mu}$$

where the term W^μ vanishes on the kernel of the Euler–Lagrange operator δL (3) of L and $U^{\nu\mu} = -U^{\mu\nu}$ is a superpotential (Theorem 6).

II. LAGRANGIAN THEORY OF EVEN AND ODD FIELDS

Lagrangian theory of even and odd fields is adequately formulated in terms of the Grassmann-graded variational bicomplex on fiber bundles and graded manifolds.^{5,8,9} In a

very general setting, let us consider a composite bundle $F \rightarrow Y \rightarrow X$ where $F \rightarrow Y$ is a vector bundle provided with bundle coordinates (x^λ, y^i, q^a) . The jet manifolds $J^r F$ of $F \rightarrow X$ also are vector bundles $J^r F \rightarrow J^r Y$ coordinated by $(x^\lambda, y_\Lambda^i, q_\Lambda^a)$, $0 \leq |\Lambda| \leq r$, where $\Lambda = (\lambda_1 \dots \lambda_k)$, $|\Lambda| = k$, denotes a symmetric multi-index. Let $(J^r Y, \mathcal{A}_r)$ be a graded manifold whose body is $J^r Y$ and whose structure ring \mathcal{A}_r of graded functions consists of sections of the exterior bundle

$$\wedge(J^r F)^* = \mathbb{R} \oplus (J^r F)^* \oplus \wedge^2(J^r F)^* \oplus \dots,$$

where $(J^r F)^*$ is the dual of $J^r F \rightarrow J^r Y$. The local odd basis for this ring is $\{c_\Lambda^a\}$, $0 \leq |\Lambda| \leq r$. Let $\mathcal{S}_r^*[F; Y]$ be the differential graded algebra (henceforth DGA) of graded differential forms on the graded manifold $(J^r Y, \mathcal{A}_r)$. The inverse system of jet manifolds $J^{r-1} Y \leftarrow J^r Y$ yields the direct system of DGAs

$$\mathcal{S}^*[F; Y] \longrightarrow \mathcal{S}_1^*[F; Y] \longrightarrow \dots \mathcal{S}_r^*[F; Y] \longrightarrow \dots$$

Its direct limit $\mathcal{S}_\infty^*[F; Y]$ is the DGA of all graded differential forms on graded manifolds $(J^r Y, \mathcal{A}_r)$. It is a $C^\infty(Y)$ -algebra locally generated by elements $(c_\Lambda^a, dx^\lambda, dy_\Lambda^i, dc_\Lambda^a)$, $0 \leq |\Lambda|$. Let us recall the formulas

$$\phi \wedge \phi' = (-1)^{|\phi||\phi'| + [\phi][\phi']} \phi' \wedge \phi, \quad d(\phi \wedge \phi') = d\phi \wedge \phi' + (-1)^{|\phi|} \phi \wedge d\phi,$$

where $[\phi]$ denotes the Grassmann parity. The collective symbol (s^A) further stands for the tuple (y^i, c^a) , called the local basis for the DGA $\mathcal{S}_\infty^*[F; Y]$. Let us denote $[A] = [s^A] = [s_\Lambda^A]$.

The DGA $\mathcal{S}_\infty^*[F; Y]$ is split into the Grassmann-graded variational bicomplex of $\mathcal{S}_\infty^0[F; Y]$ -modules $\mathcal{S}_\infty^{k,r}[F; Y]$ of r -horizontal and k -contact graded forms locally generated by the one-forms dx^λ and $\theta_\Lambda^A = ds_\Lambda^A - s_{\lambda+\Lambda}^A dx^\lambda$. This bicomplex contains the variational subcomplex

$$0 \rightarrow \mathbb{R} \longrightarrow \mathcal{S}_\infty^0[F; Y] \xrightarrow{d_H} \mathcal{S}_\infty^{0,1}[F; Y] \dots \xrightarrow{d_H} \mathcal{S}_\infty^{0,n}[F; Y] \xrightarrow{\delta} \mathcal{S}_\infty^{1,n}[F; Y], \quad (1)$$

whose coboundary operator

$$d_H(\phi) = dx^\lambda \wedge d_\lambda \phi = dx^\lambda \wedge (\partial_\lambda + \sum_{0 \leq |\Lambda|} s_{\lambda\Lambda}^A \partial_A^\Lambda) \phi, \\ d_H \circ h_0 = h_0 \circ d, \quad h_0(\theta_\Lambda^A) = 0, \quad h_0(dx^\lambda) = dx^\lambda,$$

is the total differential, and whose elements

$$L = \mathcal{L}\omega \in \mathcal{S}_\infty^{0,n}[F; Y], \quad \omega = dx^1 \wedge \dots \wedge dx^n, \quad (2)$$

$$\delta L = \theta^A \wedge \mathcal{E}_A \omega = \sum_{0 \leq |\Lambda|} (-1)^{|\Lambda|} \theta^A \wedge d_\Lambda (\partial_A^\Lambda \mathcal{L}) \omega, \quad d_\Lambda = d_{\lambda_1} \dots d_{\lambda_k}, \quad (3)$$

are graded Lagrangians and their Euler–Lagrange operators. Further, a pair $(\mathcal{S}_\infty^*[F; Y], L)$ denotes Lagrangian field theory.

Cohomology of the Grassmann-graded variational bicomplex has been obtained.^{8,10} Let us mention the following relevant results.

Theorem 1: Cohomology of the variational complex (1) equals the de Rham cohomology of a fiber bundle Y .

In particular, any odd element of this complex possesses trivial cohomology.

Theorem 2: Given a graded Lagrangian L , there is the decomposition

$$dL = \delta L - d_H \Xi_L, \quad \Xi \in \mathcal{S}_\infty^{n-1}[F; Y], \quad (4)$$

$$\Xi_L = L + \sum_{s=0} \theta_{\nu_s \dots \nu_1}^A \wedge F_A^{\lambda \nu_s \dots \nu_1} \omega_\lambda, \quad \omega_\lambda = \partial_\lambda \rfloor \omega, \quad (5)$$

$$F_A^{\nu_k \dots \nu_1} = \partial_A^{\nu_k \dots \nu_1} \mathcal{L} - d_\lambda F_A^{\lambda \nu_k \dots \nu_1} + \psi_A^{\nu_k \dots \nu_1}, \quad k = 1, 2, \dots,$$

where local graded functions ψ obey the relations

$$\psi_A^\nu = 0, \quad \psi_A^{(\nu_k \nu_{k-1}) \dots \nu_1} = 0.$$

The form Ξ_L (5) provides a global Lepage equivalent of a graded Lagrangian L . In particular, one can locally choose Ξ_L (5) where all functions ψ vanish.

The corollaries of Theorem 2 are the first variational formula (9) and the first Noether theorem (Theorem 3).

III. THE FIRST NOETHER THEOREM

In order to treat symmetries of Lagrangian field theory $(\mathcal{S}_\infty^*[F; Y], L)$ in a very general setting, we consider graded derivations of the \mathbb{R} -ring $\mathcal{S}_\infty^0[F; Y]$.^{5,8} They take the form

$$\vartheta = \vartheta^\lambda \partial_\lambda + \sum_{0 \leq |\Lambda|} \vartheta_\Lambda^A \partial_A^\Lambda, \quad \partial_A^\Lambda(s_\Sigma^B) = \partial_A^\Lambda \rfloor ds_\Sigma^B = \delta_A^B \delta_\Sigma^\Lambda. \quad (6)$$

Any graded derivation ϑ (6) yields the Lie derivative

$$\begin{aligned} \mathbf{L}_\vartheta \phi &= \vartheta \rfloor d\phi + d(\vartheta \rfloor \phi), \quad \phi \in \mathcal{S}_\infty^*[F; Y], \\ \mathbf{L}_\vartheta(\phi \wedge \sigma) &= \mathbf{L}_\vartheta(\phi) \wedge \sigma + (-1)^{[\vartheta][\phi]} \phi \wedge \mathbf{L}_\vartheta(\sigma), \end{aligned}$$

of the DGA $\mathcal{S}_\infty^*[F; Y]$.

A graded derivation ϑ (6) is called contact if the Lie derivative \mathbf{L}_ϑ preserves the ideal of contact graded forms of the DGA $\mathcal{S}_\infty^*[F; Y]$. Any contact graded derivation admits the decomposition

$$\vartheta = v_H + v_V = v^\lambda d_\lambda + [v^A \partial_A + \sum_{|\Lambda|>0} d_\Lambda (v^A - s_\mu^A v^\mu) \partial_A^\Lambda] \quad (7)$$

into the horizontal and vertical parts v_H and v_V . A glance at the expression (7) shows that a contact graded derivation ϑ is an infinite order jet prolongation of its restriction

$$v = v^\lambda \partial_\lambda + v^A \partial_A \quad (8)$$

to the graded commutative ring $S^0[F; Y]$. One calls v (8) the generalized graded vector field. It is a graded vector field if its components v^λ , v^A are independent of jets s_Λ^A . Note that generalized symmetries of Lagrangian systems have been intensively studied.^{5,11–13}

Given a contact graded derivation (7), a corollary of the decomposition (4) is the above mentioned first variational formula

$$\mathbf{L}_\vartheta L = v_V \rfloor \delta L + d_H(h_0(\vartheta \rfloor \Xi_L)) + d_V(v_H \rfloor \omega) \mathcal{L}, \quad (9)$$

where Ξ_L is the Lepage equivalent (5) of L .

Given a Lagrangian L (2), a contact graded derivation ϑ (7) is said to be its variational symmetry (strictly speaking a variational generalized supersymmetry) if the Lie derivative $\mathbf{L}_\vartheta L$ is d_H -exact, i.e.

$$\mathbf{L}_\vartheta L = d_H \sigma. \quad (10)$$

A variational symmetry ϑ of a Lagrangian L is called its exact symmetry if the Lie derivative $\mathbf{L}_\vartheta L$ vanishes.

An immediate corollary of the first variational formula (9) is the following first Noether theorem.

Theorem 3: If a contact graded derivation ϑ (7) is a variational symmetry (10) of a Lagrangian L , the first variational formula (9) restricted to $\text{Ker } \delta L$ leads to the weak conservation law

$$d_H(\sigma - h_0(\vartheta \rfloor \Xi_L)) \approx 0 \quad (11)$$

of the current

$$\mathcal{J}_\vartheta = \mathcal{J}_\vartheta^\mu \omega_\mu = \sigma - h_0(\vartheta \rfloor \Xi_L). \quad (12)$$

Obviously, the conserved current (12) is defined up to a d_H -closed horizontal $(n-1)$ -form

$$U = \frac{1}{2} U^{\nu\mu} \omega_{\nu\mu}, \quad \omega_{\nu\mu} = \partial_\nu \rfloor \omega_\mu, \quad (13)$$

called the superpotential.

Lemma 4: A glance at the expression (9) shows the following.⁵

(i) A contact graded derivation ϑ is a variational symmetry only if the generalized vector field v (8) is projected onto X , i.e., $v^\lambda \partial_\lambda$ is a vector field on X .

(ii) A contact graded derivation ϑ is a variational symmetry iff its vertical part v_V is well.

(iii) Any projectable contact graded derivation is a variational symmetry of a variationally trivial Lagrangian.

(iv) A contact graded derivation ϑ is a variational symmetry iff the graded density $v_V \rfloor \delta L$ is d_H -exact.

Variational symmetries of a Lagrangian L constitute a real vector space \mathcal{G}_L . By virtue of item (iii) of Lemma 4, the Lie superbracket

$$\mathbf{L}_{[\vartheta, \vartheta']} = [\mathbf{L}_\vartheta, \mathbf{L}_{\vartheta'}]$$

of variational symmetries is a variational symmetry. Consequently, the vector space \mathcal{G}_L of variational symmetries is a real Lie superalgebra.

By virtue of item (ii) of Lemma 4, we further restrict our consideration to vertical contact graded derivations

$$\vartheta = v^A \partial_A + \sum_{0 < |\Lambda|} d_\Lambda v^A \partial_A^\Lambda. \quad (14)$$

A graded derivation ϑ (14) is called nilpotent if $\mathbf{L}_\vartheta(\mathbf{L}_\vartheta \phi) = 0$ for any horizontal form $\phi \in \mathcal{S}_\infty^{0,*}[F; Y]$. One can show that ϑ (14) is nilpotent only if it is odd and iff $\vartheta(v) = 0$.⁵

For the sake of brevity, the common symbol v further stands for a generalized graded vector field $v = v^A \partial_A$, the vertical contact graded derivation ϑ (14) determined by v , and the Lie derivative \mathbf{L}_ϑ . We agree to call v the graded derivation of the DGA $\mathcal{S}_\infty^*[F; Y]$. The right graded derivations $\overleftarrow{v} = \overleftarrow{\partial}_A v^A$ of $\mathcal{S}_\infty^*[F; Y]$ also are considered.

IV. GAUGE SYMMETRIES

Without a loss of generality, let a Lagrangian L be even. To describe Noether identities of Lagrangian field theory $(\mathcal{S}_\infty^*[F; Y], L)$, let us introduce the following notation. Given a vector bundle $E \rightarrow X$, we call

$$\overline{E} = E^* \otimes \bigwedge^n T^* X$$

the density-dual of E . The density dual of a graded vector bundle $E = E^0 \oplus E^1$ is $\overline{E} = \overline{E}^1 \oplus \overline{E}^0$. Given a graded vector bundle $E = E^0 \oplus E^1$ over Y , we consider the composite

bundle $E \rightarrow E^0 \rightarrow X$ and denote $\mathcal{P}_\infty^*[E; Y] = \mathcal{S}_\infty^*[E; E^0]$. Let VF be the vertical tangent bundle of $F \rightarrow X$, the density-dual of the vector bundle $VF \rightarrow F$ is

$$\overline{VF} = V^*F \otimes_F^n T^*X.$$

Let us enlarge $\mathcal{S}_\infty^*[F; Y]$ to the DGA $\mathcal{P}_\infty^*[\overline{VF}; Y]$ possessing the local basis (s^A, \bar{s}_A) , $[\bar{s}_A] = ([A] + 1) \bmod 2$. Its elements \bar{s}_A are called antifields. The DGA $\mathcal{P}_\infty^*[\overline{VF}; Y]$ is endowed with the odd right graded derivation $\bar{\delta} = \overleftarrow{\partial}^A \mathcal{E}_A$, where \mathcal{E}_A are the variational derivatives (3). This graded derivation is obviously nilpotent. Then we have the chain complex

$$0 \leftarrow \text{Im } \bar{\delta} \xleftarrow{\bar{\delta}} \mathcal{P}_\infty^{0,n}[\overline{VF}; Y]_1 \xleftarrow{\bar{\delta}} \mathcal{P}_\infty^{0,n}[\overline{VF}; Y]_2 \quad (15)$$

of graded densities of antifield number ≤ 2 . Its one-cycles

$$\bar{\delta}\Phi = 0, \quad \Phi = \sum_{0 \leq |\Lambda|} \Phi^{A,\Lambda} \bar{s}_{\Lambda A} d^n x \in \mathcal{P}_\infty^{0,n}[\overline{VF}; Y]_1, \quad (16)$$

define Noether identities of Lagrangian field theory $(\mathcal{S}_\infty^*[F; Y], L)$. In particular, one-chains $\Phi \in \mathcal{P}_\infty^{0,n}[\overline{VF}; Y]_1$ are necessarily Noether identities if they are boundaries. Therefore, these Noether identities are called trivial. Accordingly, non-trivial Noether identities modulo the trivial ones are associated to elements of the first homology $H_1(\bar{\delta})$ of the complex (15).^{6,8}

Let us assume that the homology $H_1(\bar{\delta})$ is finitely generated. Namely, there exists a projective $C^\infty(X)$ -module $\mathcal{C} \subset H_1(\bar{\delta})$ of finite rank possessing the local basis $\{\Delta_r\}$ such that any element $\Phi \in H_1(\bar{\delta})$ factorizes as

$$\Phi = \sum_{0 \leq |\Xi|} G^{r,\Xi} d_\Xi \Delta_r d^n x, \quad \Delta_r = \sum_{0 \leq |\Lambda|} \Delta_r^{A,\Lambda} \bar{s}_{\Lambda A}, \quad G^{r,\Xi}, \Delta_r^{A,\Lambda} \in \mathcal{S}_\infty^0[F; Y], \quad (17)$$

through elements of \mathcal{C} . Thus, all non-trivial Noether identities (16) result from the Noether identities

$$\bar{\delta}\Delta_r = \sum_{0 \leq |\Lambda|} \Delta_r^{A,\Lambda} d_\Lambda \mathcal{E}_A = 0, \quad (18)$$

called the complete Noether identities. By virtue of the generalized Serre–Swan theorem,⁸ the module \mathcal{C} is isomorphic to the $C^\infty(X)$ -module of sections of the density-dual \overline{E} of some graded vector bundle $E \rightarrow X$.

We define a non-trivial gauge symmetry of Lagrangian field theory $(\mathcal{S}_\infty^*[F; Y], L)$ as that associated to the Noether identities (18) by means of the inverse second Noether theorem.^{6–8}

Let us enlarge the DGA $\mathcal{P}_\infty^*[\overline{VF}; Y]$ to the DGA $\mathcal{P}_\infty^*[\overline{VF} \oplus_Y E; Y]$ possessing the local basis (s^A, \bar{s}_A, c^r) . Its elements c^r of Grassmann parity $[c_r] = [\Delta_r]$ are called the ghosts. The

graded derivation $\bar{\delta}$ is naturally prolonged to the DGA $\mathcal{P}_\infty^*[\overline{VF} \oplus_Y E; Y]$. Let us extend an original Lagrangian L to the even Lagrangian

$$L_e = L + c^r \Delta_r \omega \in \mathcal{P}_\infty^{0,n}[\overline{VF} \oplus_Y E; Y]. \quad (19)$$

It is readily observed that, by virtue of the Noether identities (18), the graded derivation $\bar{\delta}$ is an exact symmetry of L_e (19). It follows from item (iv) of Lemma 4 that

$$\frac{\bar{\delta}(c^r \Delta_r)}{\delta \bar{s}_A} \mathcal{E}_A \omega = \sum_{0 \leq |\Lambda|} (-1)^{|\Lambda|} d_\Lambda (c^r \Delta_r^{A,\Lambda}) \mathcal{E}_A \omega = u^A \mathcal{E}_A \omega = d_H \sigma. \quad (20)$$

Then by the same reason, the odd graded derivation

$$u = u^A \frac{\partial}{\partial s^A}, \quad u^A = \sum_{0 \leq |\Lambda|} c_\Lambda^r \eta(\Delta_r^A)^\Lambda, \quad (21)$$

of $\mathcal{P}_\infty^*[\overline{VF}; Y]$ is a variational symmetry of an original Lagrangian L .

A glance at the expression (21) shows that the variational symmetry u is a linear differential operator on the $C^\infty(X)$ -module \mathcal{C} of ghosts with values in the real space \mathcal{G}_L of variational symmetries. It is called the gauge symmetry of a Lagrangian L which is associated to the complete non-trivial Noether identities (18).

This association is unique due to the following. The variational derivative of the equality (20) with respect to ghosts c^r leads to the equalities

$$\delta_r(u^A \mathcal{E}_A \omega) = \sum_{0 \leq |\Lambda|} (-1)^{|\Lambda|} d_\Lambda (\eta(\Delta_r^A)^\Lambda \mathcal{E}_A) = \sum_{0 \leq |\Lambda|} \eta(\eta(\Delta_r^A))^\Lambda d_\Lambda \mathcal{E}_A = \sum_{0 \leq |\Lambda|} \Delta_r^{A,\Lambda} d_\Lambda \mathcal{E}_A = 0$$

which reproduce the complete non-trivial Noether identities (18).

Moreover, the gauge symmetry u (21) is complete in the following sense. Let

$$\sum_{0 \leq |\Xi|} C^R G_R^{r,\Xi} d_\Xi \Delta_r \omega$$

be some projective $C^\infty(X)$ -module of finite rank of non-trivial Noether identities parameterized by the corresponding ghosts C^R . A direct computation shows that the graded derivation

$$d_\Lambda \left(\sum_{0 \leq |\Xi|} \eta(G_R^r)^\Xi C_\Xi^R \right) u_r^{A,\Lambda} \frac{\partial}{\partial s^A}$$

is a variational symmetry of a Lagrangian L and, consequently, its gauge symmetry parameterized by ghosts C^R .^{7,8} It factorizes through the gauge symmetry (21) by putting ghosts

$$c^r = \sum_{0 \leq |\Xi|} \eta(G_R^r)^\Xi C_\Xi^R.$$

Thus, we come to the following second Noether theorem.

Theorem 5: The odd graded derivation u (21) is a complete non-trivial gauge symmetry of a Lagrangian L associated to the complete non-trivial Noether identities (18).

V. GAUGE CONSERVATION LAWS

Being a variational symmetry, the gauge symmetry u (21) defines the weak conservation law (11). The peculiarity of this conservation law is that the conserved current \mathcal{J}_u (12) is reduced to a superpotential as follows.

Theorem 6: If u (21) is a gauge symmetry of a Lagrangian L , the corresponding conserved current \mathcal{J}_u (12) takes the form

$$\mathcal{J}_u = W + d_H U = (W^\mu + d_\nu U^{\nu\mu})\omega_\mu, \quad (22)$$

where the form W is $\bar{\delta}$ -exact (i.e. it vanishes on-shell) and U is a superpotential (13).

Proof: Let the gauge symmetry u (21) be at most of jet order N in ghosts. Then the conserved current \mathcal{J}_u (22) is decomposed into the sum

$$\begin{aligned} \mathcal{J}_u^\mu = & J_r^{\mu\mu_1\cdots\mu_M} c_{\mu_1\cdots\mu_M}^r + \sum_{1 \leq k < M} J_r^{\mu\mu_k\cdots\mu_M} c_{\mu_k\cdots\mu_M}^r + \\ & J_r^{\mu\mu_M} c_{\mu_M}^r + J_r^\mu c^r + J^\mu, \quad N \leq M, \end{aligned} \quad (23)$$

and the first variational formula (9) takes the form

$$\begin{aligned} 0 = & \left[\sum_{k=1}^N u_{V_r}^{i\mu_k\cdots\mu_N} c_{\mu_k\cdots\mu_N}^r + u_{V_r}^i c^r \right] \mathcal{E}_i - \\ & d_\mu \left(\sum_{k=1}^M J_r^{\mu\mu_k\cdots\mu_M} c_{\mu_k\cdots\mu_M}^r + J_r^\mu c^r + J^\mu \right). \end{aligned}$$

This equality provides the following set of equalities for each $c_{\mu\mu_1\cdots\mu_M}^r$, $c_{\mu_k\cdots\mu_M}^r$ ($k = 1, \dots, M - N - 1$), $c_{\mu_k\cdots\mu_N}^r$ ($k = 1, \dots, N - 1$), c_μ^r and c^r :

$$0 = J_r^{(\mu\mu_1)\cdots\mu_M}, \quad (24)$$

$$0 = J_r^{(\mu_k\mu_{k+1})\cdots\mu_M} + d_\nu J_r^{\nu\mu_k\cdots\mu_M}, \quad 1 \leq k < M - N, \quad (25)$$

$$0 = u_{V_r}^{i\mu_k\cdots\mu_N} \mathcal{E}_i - J_r^{(\mu_k\mu_{k+1})\cdots\mu_N} - d_\nu J_r^{\nu\mu_k\cdots\mu_N}, \quad 1 \leq k < N, \quad (26)$$

$$0 = u_{V_r}^i \mathcal{E}_i - J_r^\mu - d_\nu J_r^{\nu\mu}, \quad (27)$$

where $(\mu\nu)$ means symmetrization of indices in accordance with the splitting

$$J_r^{\mu_k\mu_{k+1}\cdots\mu_N} = J_r^{(\mu_k\mu_{k+1})\cdots\mu_N} + J_r^{[\mu_k\mu_{k+1}]\cdots\mu_N}.$$

We also have the equalities

$$0 = u_{V_r}^i \mathcal{E}_i - d_\mu J_r^\mu, \quad (28)$$

$$0 = d_\mu J^\mu. \quad (29)$$

With the equalities (24) – (27), the decomposition (23) takes the form

$$\begin{aligned} \mathcal{J}_u^\mu &= J_r^{[\mu\mu_1]\dots\mu_M} c_{\mu_1\dots\mu_M}^r + \\ &\quad \sum_{1 \leq k \leq M-N} [(J_r^{[\mu\mu_k]\dots\mu_M} - d_\nu J_r^{\nu\mu\mu_k\dots\mu_M}) c_{\mu_k\dots\mu_M}^r] + \\ &\quad \sum_{1 \leq k \leq N} [(u_{V_r}^{i\mu\mu_k\dots\mu_N} \mathcal{E}_i - d_\nu J_r^{\nu\mu\mu_k\dots\mu_N} + J_r^{[\mu\mu_k]\dots\mu_N}) c_{\mu_k\dots\mu_N}^r] + \\ &\quad (u_{V_r}^{i\mu\mu_N} \mathcal{E}_i - d_\nu J_r^{\nu\mu\mu_N} + J_r^{[\mu\mu_N]}) c_{\mu_N}^r + (u_{V_r}^{i\mu} \mathcal{E}_i - d_\nu J_r^{\nu\mu}) c^r + J^\mu. \end{aligned}$$

A direct computation

$$\begin{aligned} \mathcal{J}_u^\mu &= d_\nu (J_r^{[\mu\nu]\mu_2\dots\mu_M} c_{\mu_2\dots\mu_M}^r) - d_\nu J_r^{[\mu\nu]\mu_2\dots\mu_M} c_{\mu_2\dots\mu_M}^r + \\ &\quad \sum_{1 \leq k \leq M-N} [d_\nu (J_r^{[\mu\nu]\mu_{k+1}\dots\mu_M} c_{\mu_{k+1}\dots\mu_M}^r) - \\ &\quad d_\nu J_r^{[\mu\nu]\mu_{k+1}\dots\mu_M} c_{\mu_{k+1}\dots\mu_M}^r - d_\nu J_r^{\nu\mu\mu_k\dots\mu_M} c_{\mu_k\dots\mu_M}^r] + \\ &\quad \sum_{1 \leq k \leq N} [(u_{V_r}^{i\mu\mu_k\dots\mu_N} \mathcal{E}_i - d_\nu J_r^{\nu\mu\mu_k\dots\mu_N}) c_{\mu_k\dots\mu_N}^r + \\ &\quad d_\nu (J_r^{[\mu\nu]\mu_{k+1}\dots\mu_N} c_{\mu_{k+1}\dots\mu_N}^r) - d_\nu J_r^{[\mu\nu]\mu_{k+1}\dots\mu_N} c_{\mu_{k+1}\dots\mu_N}^r] + \\ &\quad [(u_{V_r}^{i\mu\mu_N} \mathcal{E}_i - d_\nu J_r^{\nu\mu\mu_N}) c_{\mu_N}^r + d_\nu (J_r^{[\mu\nu]} c^r) - d_\nu J_r^{[\mu\nu]} c^r] + \\ &\quad (u_{V_r}^{i\mu} \mathcal{E}_i - d_\nu J_r^{\nu\mu}) c^r + J^\mu \\ &= d_\nu (J_r^{[\mu\nu]\mu_2\dots\mu_M} c_{\mu_2\dots\mu_M}^r) + \\ &\quad \sum_{1 \leq k \leq M-N} [d_\nu (J_r^{[\mu\nu]\mu_{k+1}\dots\mu_M} c_{\mu_{k+1}\dots\mu_M}^r) - d_\nu J_r^{(\nu\mu)\mu_k\dots\mu_M} c_{\mu_k\dots\mu_M}^r] + \\ &\quad \sum_{1 \leq k \leq N} [(u_{V_r}^{i\mu\mu_k\dots\mu_N} \mathcal{E}_i - d_\nu J_r^{(\nu\mu)\mu_k\dots\mu_N}) c_{\mu_k\dots\mu_N}^r + \\ &\quad d_\nu (J_r^{[\mu\nu]\mu_{k+1}\dots\mu_N} c_{\mu_{k+1}\dots\mu_N}^r)] + \\ &\quad [(u_{V_r}^{i\mu\mu_N} \mathcal{E}_i - d_\nu J_r^{(\nu\mu)\mu_N}) c_{\mu_N}^r + d_\nu (J_r^{[\mu\nu]} c^r)] + (u_{V_r}^{i\mu} \mathcal{E}_i - d_\nu J_r^{(\nu\mu)}) c^r + J^\mu \end{aligned}$$

leads to the expression

$$\begin{aligned} \mathcal{J}_u^\mu &= \left(\sum_{1 \leq k \leq N} u_{V_r}^{i\mu\mu_k\dots\mu_N} c_{\mu_k\dots\mu_N}^r + u_{V_r}^{i\mu} c^r \right) \mathcal{E}_i - \\ &\quad \left(\sum_{1 \leq k \leq M} d_\nu J_r^{(\nu\mu)\mu_k\dots\mu_M} c_{\mu_k\dots\mu_M}^r + d_\nu J_r^{(\nu\mu)} c^r \right) - \end{aligned} \quad (30)$$

$$d_\nu \left(\sum_{1 < k \leq M} J^{[\nu\mu]\mu_k \dots \mu_M} c_{\mu_k \dots \mu_M}^r + J_r^{[\nu\mu]} c^r \right) + J^\mu.$$

The first summand of this expression vanishes on-shell. Its second one contains the terms $d_\nu J^{(\nu\mu_k)\mu_{k+1} \dots \mu_M}$, $k = 1, \dots, M$. By virtue of the equalities (25) – (26), every $d_\nu J^{(\nu\mu_k)\mu_{k+1} \dots \mu_M}$ is expressed into the terms vanishing on-shell and the term $d_\nu J^{(\nu\mu_{k-1})\mu_k \dots \mu_M}$. Iterating the procedure and bearing in mind the equality (24), one can easily show that the second summand of the expression (30) also vanishes on-shell. Finally, the condition (29) means that the odd $(n-1)$ -form $J^\mu \omega_\mu$ is d_H -closed and, consequently, it is d_H -exact in accordance with Theorem 1. Thus, the current \mathcal{J}_u takes the form (22).

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